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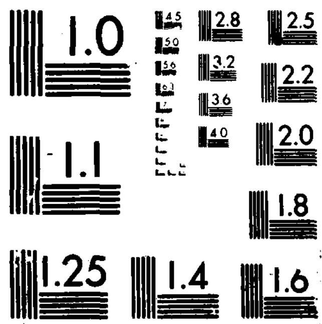


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This is an expository survey of the (univariate) theory of extreme values and the estimation of the parameters of the related extreme-value distributions. It will appear in the forthcoming *Encyclopedia of Statistical Sciences* to be published by John Wiley and Sons, Inc.

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*Abstract*

This is an expository survey of the (univariate) theory of extreme values and the estimation of the parameters of the related extreme-value distributions. This survey was written at the invitation of Professors N.L. Johnson and S. Kotz, editors of the forthcoming *Encyclopedia of Statistical Sciences* to be published by John Wiley and Sons, Inc., New York. The theory of extreme values plays a fundamental role in several areas of applied statistics, such as the analysis of flood flows, the reliability of complex systems, the analysis of air pollution data, etc. In addition to surveying (without proofs) the basic results of extreme value theory and the estimators of the parameters of the extreme value distributions, this survey presents a brief discussion of the current research in these areas.

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# EXTREME VALUE DISTRIBUTIONS

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The theory of extreme values, and the extreme value distributions, play an important role in theoretical and applied statistics. For example, extreme value distributions arise quite naturally in the study of size effect on material strengths, the occurrence of floods and droughts, the reliability of systems made up of a large number of components, and in assessing the levels of air pollution. Other applications of extreme value distributions arise in the study of what are known as "record values" and "breaking records." For an up-to-date and a fairly complete reference on the theory of extreme values, we refer the reader to the recent book by Galambos (1978). For a more

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classical yet honorable treatise on the subject, we refer to Gumbel (1958).

## 1. Preliminaries

Suppose that $X_1, X_2, \cdots, X_n$ are independent and identically distributed random variables from a distribution $F(x)$ which is assumed to be continuous. The theory of extreme values primarily concerns itself with the distribution of the smallest and largest values of $X_1, X_2, \cdots, X_n$. That is, if

$$X_{1:n} = \min(X_1, X_2, \cdots, X_n) = X_{(1)} \tag{1.1}$$

and

$$X_{n:n} = \max(X_1, X_2, \cdots, X_n) = X_{(n)} , \tag{1.2}$$

then knowing $F(x)$, we would like to say something about $L_n(x) = \Pr\left[X_{(1)} \leq x\right]$ and $H_n(x) = \Pr\left[X_{(n)} \leq x\right]$. The random variables $X_{(1)}$ and $X_{(n)}$ are also known as the extreme values.

In order to give some motivation as to why the random variables, $X_{(1)}$ and $X_{(n)}$, and their distribution functions are of interest to us, we shall consider the following situations:

1. Consider a chain which is made up of n links; the chain breaks when any one of its links break. The first link to break is the weakest link; that is, the one which has the smallest strength. It is meaningful to assume that the strength of the $i^{th}$ link, say $X_i$, $i=1,2,\cdots,n$ is a random variable with distribution function $F(x)$. Since the chain breaks when its weakest link fails, the strength of the chain is therefore described by the random variable $X_{(1)} = \min(X_1, X_2, \cdots, X_n)$.

2. Consider an engineering or a biological system which is made up of n identical components, all of which may function simultaneously. For example, a large airplane may contain four identical engines which could be functioning simultaneously, or the human respiratory system which consists of two identical lungs. The system functions as long as any one of the n components is functioning. Such systems are known as *parallel-redundant systems* and occur quite often in practice. Suppose that the time to failure (the lifelength) of the $i^{th}$ component, say $X_i$, $i=1,2,\cdots,n$ is a random variable with distribution function $F(x)$. Since the system fails at the time of failure of the last component, the lifelength of the system is described by the random variable $X_{(n)} = \max(X_1,X_2,\cdots,X_n)$.

It is easy to envision several other physical situations in which the random variables $X_{(1)}$ and $X_{(n)}$ arise quite naturally. For instance, the use of $X_{(n)}$ for setting air pollution standards is discussed by Singpurwalla (1972) and by Mittal (1978); and the use of $X_{(1)}$ in studying the time for a liquid to corrode through a surface having a large number of small pits is discussed in Mann, Schafer, and Singpurwalla (1974), p. 130.

## 2. Distribution of the Extreme Values

Even though our assumption that $X_1,X_2,\cdots,X_n$ are independent is hard to justify in practice, we shall, in the interest of simplicity and an easier exposition, continue to retain it. Note that

$$L_n(x) = \Pr\left[X_{(1)} < x\right] = 1 - \Pr\left[X_{(1)} > x\right]$$

$$= 1 - \Pr\left[X_1 > x, X_2 > x, \cdots, X_n > x\right] ,$$

since the probability that the smallest value is larger than x is the same as the probability that all the n observations exceed x. Because of independence

$$L_n(x) = 1 - \prod_{i=1}^{n} \Pr\left[X_i > x\right] = 1 - (1 - F(x))^n , \qquad (2.1)$$

since all the n observations have a common distribution F(x). Using analogous arguments we can show that

$$H_n(x) = \Pr\left[X_{(n)} \leq x\right] = (F(x))^n . \qquad (2.2)$$

Thus under independence, when F(x) is completely specified, we can, in principle, find the distribution of $X_{(1)}$ and $X_{(n)}$. Often the distribution functions, $L_n(x)$ and $H_n(x)$, take simple forms. For example, if F(x) is an exponential distribution with a scale parameter $\lambda > 0$, that is, if $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$, then $L_n(x) = 1 - e^{-n\lambda x}$ — again an exponential distribution with a scale parameter $n\lambda$.

Despite the simplicity of the above results, there are two considerations which motivate us to going beyond Equations (2.1) and (2.2). The first consideration pertains to the fact that in many cases $L_n(x)$ or $H_n(x)$ do not take simple and manageable forms, and the second consideration is motivated by the fact that in many practical applications of the extreme value theory n is very large. For example, if $F(x) = 1 - e^{-\lambda x}$, then $H_n(x) = \left(1 - e^{-\lambda x}\right)^n$, and when F(x) is the distribution function of a standard normal variate, then

$H_n(x) = \left(\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds\right)^n$. It so happens that under some very general conditions on F(x), the distributions of $X_{(1)}$ and $X_{(n)}$ when n becomes large take simple forms. The distributions $L_n(x)$ and $H_n(x)$, when $n \to \infty$, are known as the *asymptotic (or the limiting) distribution*

*of extreme values*, and the associated theory which enables us to study these is known as the *asymptotic theory of extremes*; the word "asymptotic" describes the fact that n is getting large.

## 2.1 The Asymptotic Distribution of Extremes

The key notion which makes the asymptotic distributions of $X_{(1)}$ and $X_{(n)}$ of interest is that for some constants $\alpha_n$, $\beta_n > 0$, $\gamma_n$, and $\delta_n > 0$, the quantities $(X_{(1)} - \alpha_n)/\beta_n$ and $(X_{(n)} - \gamma_n)/\delta_n$ become more and more independent of n. The $\alpha_n$, $\beta_n$, $\gamma_n$, and $\delta_n$ are referred to as the *normalizing constants*. A goal of the asymptotic theory of extreme values is to specify the conditions under which the normalizing constants exist, and to determine the limiting distribution functions L(x) and H(x) where

$$\lim_{n\to\infty} \Pr\left[\frac{X_{(1)} - \alpha_n}{\beta_n} < x\right] = \lim_{n\to\infty} L_n(\alpha_n + \beta_n x) = L(x) \qquad (2.3)$$

and

$$\lim_{n\to\infty} \Pr\left[\frac{X_{(n)} - \gamma_n}{\delta_n} < x\right] = \lim_{n\to\infty} H_n(\gamma_n + \delta_n x) = H(x) \ . \qquad (2.4)$$

$$\max[\min](X_1, X_2, \cdots, X_n) = -\min[\max](-X_1, -X_2, \cdots, -X_n) \ , \qquad (2.5)$$

the theory for the largest extreme is identical to the theory for the smallest extreme and vice versa. However, we shall, for the sake of completeness, give the pertinent results for both the maxima and the minima.

The fundamental result in the theory of extreme values was discovered by Frechet, and by Fisher and Tippet in 1928, and was formalized in 1943 by Gnedenko. It states that if $(X_{(n)} - \gamma_n)/\delta_n$ has a limiting distribution H(x), then H(x) *must have one of the three possible forms*. An analogous result also holds for $(X_{(1)} - \alpha_n)/\beta_n$. The immediate implication of this result is that irrespective of what the original distribution F is, the asymptotic distribution of $X_{(n)}$ (if it exists) is any one of three possible forms. Thus, the asymptotic distribution of the extreme values is in some sense akin to the normal distribution for the sample mean. This property of the asymptotic distribution of the extremes is another motivation for our study of the limiting distributions.

We shall summarize the above results via the following theorem of Gnedenko.

Theorem 2.1 (Gnedenko): Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed with distribution function F, and let $X_{(n)} = \max(X_1, X_2, \cdots, X_n)$. Suppose that for some sequences of normalizing constants $\{\gamma_n\}$, and $\{\delta_n > 0\}$, and some other constants $a \geq 0$, $b > 0$

$$\lim_{n \to \infty} \Pr \left\{ \frac{X_{(n)} - \gamma_n}{\delta_n} \leq \frac{x - a}{b} \right\} = H\left(\frac{x - a}{b}\right) \tag{2.6}$$

for all continuity points of x, where H(·) is a nondegenerate distribution function. Then, H(·) must belong to one of the following three "extreme value types":

$$\text{I(largest)} \qquad H^{(1)}\left(\frac{x-a}{b}\right) = \exp\left(-\exp\left(-\frac{x-a}{b}\right)\right), \quad -\infty < x < \infty, \tag{2.7}$$

$$\text{II(largest)} \quad H^{(2)}\left(\frac{x-a}{b}\right) = \begin{cases} 0 & , x<a \\ \exp\left(-\left(\frac{x-a}{b}\right)^{-\alpha}\right) & , x \geq a \ , \ \alpha>0 \ , \end{cases} \tag{2.8}$$

$$\text{III(largest)} \quad H^{(3)}\left(\frac{x-a}{b}\right) = \begin{cases} \exp\left(-\left(-\frac{x-a}{b}\right)^{\alpha}\right) & , x \leq a \ , \ \alpha>0 \\ 1 & , x \geq a \ . \end{cases} \tag{2.9}$$

Whenever Equation (2.7), or (2.8), or (2.9) holds for some sequences $\{\gamma_n\}$ and $\{\delta_n > 0\}$, we shall say that F belongs to the *domain of attraction* of $H^{(i)}$, i=1,2, or 3, and write $F \in \mathcal{D}(H^{(i)})$. Furthermore, it is not necessary for us to know the exact form of F in order to determine to which domain of attraction it belongs. A useful feature of the extreme value theory is that it is just the behavior of the tail of F(x) that determines its domain of attraction. Thus, a good deal can be said about the asymptotic behavior of $X_{(n)}$ based on a limited knowledge about F. We shall formalize the above facts by giving below the necessary and sufficient conditions for $F \in \mathcal{D}(H^{(i)})$, i=1,2,3.

Theorem 2.2 (Gnedenko): Let $x_o \leq \infty$ be such that $F(x_o) = 1$, and $F(x) < 1$ for all $x < x_o$. Then

a) $F \in \mathcal{D}(H^{(1)})$ if and only if there exists a continuous function $A(x)$ such that $\lim_{x \to x_o} A(x) = 0$, and such that for all h,

$$\lim_{x \to x_o} \frac{1 - F\{x(1 + hA(x))\}}{1 - F(x)} = e^{-h} \ ;$$

b) $F \in \mathcal{D}(H^{(2)})$ if and only if $\lim_{x \to \infty} \frac{1 - F(x)}{1 - F(kx)} = k^{\alpha}$ for each $k>0$, and $\alpha>0$;

c) $F \in \mathscr{D}(H^{(3)})$ if and only if $\lim_{h \to 0} \frac{1 - F(x_0 - kh)}{1 - F(x_0 - h)} = k^{\alpha}$ for each $k > 0$, and $\alpha > 0$.

We note from the above theorem, the role played by $x_0$, the tail point of F; that is, the point where $F = 1$.

Using the criteria given in Theorem 2.2, we can verify that if F is either an exponential, or a normal, or a Weibull distribution ($F(x) = 1 - \exp(-x^{\alpha})$, $x \geq 0$, $\alpha > 0$), then F belongs to the domain of attraction of $H^{(1)}$ whereas if F is a uniform distribution, then $F \in \mathscr{D}(H^{(3)})$; this conclusion of course is for the largest values. Another property exhibited by the extreme value type distributions $H^{(i)}(\cdot)$, i=1,2,3, is that they belong to their own domain of attraction. That is, $H^{(i)} \in \mathscr{D}(H^{(i)})$, for i=1,2, or 3; this is also referred to as the *self-locking property*.

Methods for determining the constants $\gamma_n$ and $\delta_n$ involve some additional notation and detail, and these can be found in Gnedenko (1943) or Calambos (1978).

Analogous to the three "extreme value types" for the largest values given in Theorem 2.1, we have three extreme value types for the smallest values $X_{(1)}$. That is, if

$\lim_{n \to \infty} \Pr\left\{\frac{X_{(1)} - \alpha_n}{\beta_n} < \frac{x - a}{b}\right\} = L\left(\frac{x - a}{b}\right)$, then $L(\cdot)$ must belong to one of the following:

$$\text{I(smallest)} \qquad L^{(1)}\left(\frac{x - a}{b}\right) = 1 - \exp\left(-\exp\left(\frac{x - a}{b}\right)\right) \quad , \quad -\infty < x < \infty \; , \qquad (2.10)$$

$$\text{II(smallest)} \quad L^{(2)}\left(\frac{x-a}{b}\right) = \begin{cases} 0 & , \quad x<a \\ 1-\exp\left(-\left(\frac{x-a}{b}\right)^{\alpha}\right) & , \quad x\geq a \ , \ \alpha>0 \ , \end{cases} \tag{2.11}$$

$$\text{III(smallest)} \quad L^{(3)}\left(\frac{x-a}{b}\right) = \begin{cases} 1-\exp\left(-\left(-\frac{x-a}{b}\right)^{-\alpha}\right) & , \quad x<a \ , \ \alpha>0 \ , \\ 1 & , \quad x\geq a \ . \end{cases} \tag{2.12}$$

Using criteria which are analogous to Theorem 2.2, we can verify that if F is a normal distribution, then $F\in\mathcal{D}(L^{(1)})$, whereas if F is an exponential, a uniform, or a Weibull, then $F\in\mathcal{D}(L^{(2)})$. Here again, the distributions $L^{(i)}$ are self-locking. By way of a comment, we note that $L^{(2)}(\cdot)$ is in fact the Weibull distribution which was mentioned before and which is quite popular in reliability theory.

Current research in extreme value theory is being vigorously pursued from the point of view of dropping the assumption of independence and considering dependent sequences $X_1,X_2,\cdots,X_n$. One widely used class of dependent random variables is the *exchangeable* one.

Definition [Galambos (1978), p. 127]: The random variables $X_1,X_2,\cdots,X_n$ are said to be exchangeable if the distribution of the vector $(X_{i_1},X_{i_2},\cdots,X_{i_n})$ is identical to that of $(X_1,\cdots,X_n)$ for all permutations $(i_1,i_2,\cdots,i_n)$ of the subscripts $(1,2,\cdots,n)$.

Generalizations of Gnedenko's results when the sequence $X_1,\cdots,X_n$ is exchangeable are given in Chapter 3 of Galambos (1978). For an excellent and a very readable, albeit mathematical, survey of results when the sequence $X_1,\cdots,X_n$ is dependent, we refer the reader to Leadbetter (1975).

Another aspect of the current research in extreme value theory pertains to multivariate extreme value distributions. An entry on "Multivariate Extreme Value Distributions" appears in the forthcoming *Encyclopedia of Statistical Sciences*.

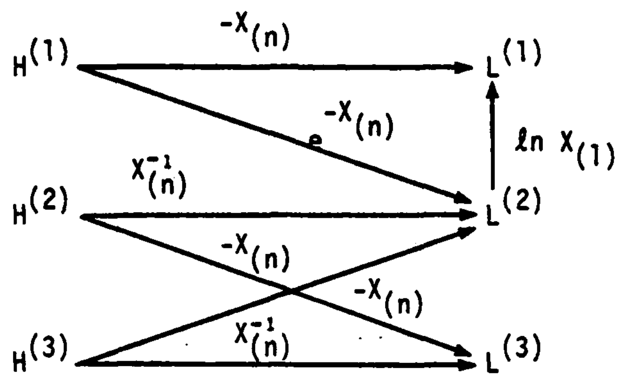
## 3. Estimation of the Parameters of the Asymptotic Distributions

In order for us to discuss methods for estimating the parameters a, b, and $\alpha$ of the distributions $H^{(i)}$ and the $L^{(i)}$, i=1,2,3, it will be helpful if we recognize several relationships which exist between them.

For example, if we denote the asymptotic distribution of $X_{(n)}$, $H^{(1)}\left(\frac{x-a}{b}\right)$ by $H^{(1)}(a,b)$, and the asymptotic distribution of $X_{(1)}$, $L^{(1)}\left(\frac{x-a}{b}\right)$ by $L^{(1)}(a,b)$, then it can be verified that $Y_{(n)} \overset{def}{=} -X_{(n)}$, has the distribution $L^{(1)}(-a,b)$. We shall denote the above relationship by writing " $H^{(1)}(a,b) \xrightarrow{-X_{(n)}} L^{(1)}(-a,b)$. " In a similar manner, if we denote $H^{(i)}\left(\frac{x-a}{b}\right)$ and $L^{(i)}\left(\frac{x-a}{b}\right)$ by $H^{(i)}(a,b,\alpha)$ and $L^{(i)}(a,b,\alpha)$ respectively, for i=2,3, then $H^{(2)}(a,b,\alpha) \xrightarrow{-X_{(n)}} L^{(3)}(-a,b,\alpha)$ and $H^{(3)}(a,b,\alpha) \xrightarrow{-X_{(n)}} L^{(2)}(-a,b,\alpha)$. If, however, $Y_{(n)} \overset{def}{=} X_{(n)}^{-1}$, and the location parameter $a=0$, then $H^{(2)}(0,b,\alpha) \xrightarrow{X_{(n)}^{-1}} L^{(2)}(0,b^{-1},\alpha)$ and $H^{(3)}(0,b,\alpha) \xrightarrow{X_{(n)}^{-1}} L^{(3)}(0,b^{-1},\alpha)$. Other transformations that are of interest are $Y_{(n)} = e^{-X_{(n)}}$ and $Y_{(1)} = \ell n\, X_{(1)}$; these give us $H^{(1)}(a,b) \xrightarrow{e^{-X_{(n)}}} L^{(2)}(0,e^{-a},b^{-1})$ and $L^{(2)}(0,b,\alpha) \xrightarrow{\ell n\, X_{(1)}} L^{(1)}(\ell n\, b,\alpha^{-1})$.

If we suppress the arguments of the $H^{(i)}$ and the $L^{(i)}$, i=1,2,3, then the following illustration, suggested to us by Mr. M.Y. Wong, is a convenient summary of the above relationships.

It is easy to verify that in the following illustration the reverse relationships also hold. For example, if




$Y_{(n)} \stackrel{\text{def}}{=} -X_{(1)}$, then $L^{(3)}(a,b,\alpha) \xrightarrow{-X_{(1)}} H^{(2)}(-a,b,\alpha)$, and so on. In view of this last relationship, and the relationships implied by the illustration given above, it follows that we need only consider the distribution $L^{(1)}(a,b)$. All the other distributions considered here can be transformed to the distribution $L^{(1)}(a,b)$, either by a change of variable or by a change of variable with a setting of the location parameter equal to zero. It is because of this fact that some of the literature on the Weibull distribution with a location parameter of 0 ($L^{(2)}(0,b,\alpha)$) appears under the heading of "an extreme value distribution" which is a common way of referring to the distribution $L^{(1)}(\cdot,\cdot)$.

When the location parameter a associated with the distributions $H^{(i)}$ and $L^{(i)}$, $i=2,3$, cannot be set equal to zero, most of the relationships mentioned before do not hold, and thus we cannot be content by just considering the distribution $L^{(1)}(a,b)$. We will have to consider both $H^{(2)}(a,b,\alpha)$ and $H^{(3)}(a,b,\alpha)$ or their duals $L^{(3)}(-a,b,\alpha)$ and $L^{(2)}(-a,b,\alpha)$, respectively. Estimation of the parameters a (or -a), b, and $\alpha$ is discussed in the next section.

### 3.1 Estimation for the Three-Parameter Distributions

The standard approach for estimating the three parameters associated with $H^{(i)}$ and $L^{(i)}$, i=2,3, is the one based on the method of maximum likelihood. Because of the popularity of the Weibull distribution, the case $L^{(2)}(a,b,\alpha)$ has been investigated very extensively. We shall give below an outline of the results for this case, and guide the reader to the relevant references.

Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the smallest ordered observations in a sample of size n from the distribution $L^{(2)}(a,b,\alpha)$. Harter and Moore (1965), and also Mann, Schafer, and Singpurwalla (1974), p. 186, (to be henceforth abbreviated as MSS), give the three likelihood equations and suggest procedures for an iterative solution of these. They also give suggestions for dealing with problems which arise when the likelihood function increases monotonically in $(0,X_{(1)})$.

Lemon (1974) modified the likelihood equations so that one need iteratively solve only two equations for estimates of the location parameter a and the shape parameter $\alpha$, which then specify an estimate of the scale parameter b.

MSS discuss, as well, the graphical method of estimation, quick initial estimates proposed by Dubey (1966), and iterative procedures involving linear estimates as leading to a median unbiased estimate of a. (A recent result of Somerville (1977) suggests that in iteratively obtaining a median unbiased estimate of a Weibull location parameter, k, defined at the bottom of p. 341 in MSS, should be approximately k/5.)

Rockette, Antle, and Klimko (1974) have conjectured that there are never more than two solutions to the likelihood equations. They show that if there exists a solution that is a local maximum, there is a second solution that is a saddle point. They also show that, even if a solution $(\hat{a},\hat{b},\hat{\alpha})$ is a local maximum, the value of the likelihood

function $L(\hat{a},\hat{b},\hat{\alpha})$ may be less than $L(a_0,b_0,\alpha_0)$ where $a_0 = x_{(1)}$, and $\alpha_0 = 1$, and $b_0$ = maximum likelihood estimate of the mean of a two-parameter exponential distribution.

### 3.2 Estimation for the Two-Parameter Distributions

When the location parameter a associated with the distributions $H^{(i)}$ and $L^{(i)}$, i=2,3, is known, or can be set equal to zero, then there are several approaches that can be used to obtain good point estimators of the parameters b and $\alpha$. The same is also true when we are interested in the parameters a and b of $H^{(1)}$ and $L^{(1)}$. These approaches involve an iterative solution of the maximum likelihood equations, and the use of linear estimation techniques.

#### 3.2.1 Maximum Likelihood Estimation

The maximum likelihood method has the advantage that it can be applied efficiently to any sort of censoring of the data.

For all the extreme-value distributions, the order statistics are the sufficient statistics. Thus, unless there are only two observations, the sufficient statistics are not complete and no small-sample optimality properties hold for the maximum likelihood estimators. The maximum likelihood estimators of the two parameters are, however, asymptotically unbiased as well as asymptotically normal and asymptotically efficient. One can use the maximum likelihood estimates with tables of Thoman, Bain, and Antle (1970) and of Billman, Antle, and Bain (1971) to obtain confidence bounds on the parameters.

#### 3.2.2 Linear Estimation Techniques

Linear techniques allow for the estimation of the two parameters of interest without the necessity of iteration. See MSS pp. 191-220

and the entries Weibull distribution, *best linear invariant* (BLI) estimators and *best linear unbiased* (BLU) estimators. Tables of Mann, Fertig, and Scheuer (1971) and Mann and Fertig (1973) can be used with either the BLI or BLU estimates to obtain confidence and tolerance bounds for censored samples of size n, n=3(1)25. See also MSS, p. 222, for tables with n=3(1)13. Thomas and Wilson (1972) compare the BLU and BLI estimators with other approximately optimal linear estimators based on all the order statistics.

If samples are complete and sample sizes are rather large, one can use tables of Chan and Kabir (1969) or of Hassanein (1972) to obtain linear estimates of a and b based on from 2 through 10 order statistics. These tables apply to weights and spacings for the order statistics that define estimators that are asymptotically unbiased with asymptotically smallest variance. Hassanein's results have the restriction that the spacings are the same for both estimators, but he also considers samples with 10 percent censoring. Tables of Mann and Fertig (1977) allow for removal of small-sample bias from Hassanein's estimators and give exact variances and covariances. This enables one to calculate approximate confidence bounds from these estimators.

For samples having only the first r of n possible observations, the unbiased linear estimator of Engelhardt and Bain (1973) for the parameter b, $b^{**}_{r,n} = \sum_{i=1}^{r} |X_{(s)} - X_{(i)}|(nk_{r,n})^{-1}$ is very efficient, especially for heavy censoring. To obtain $b^{**}_{r,n}$, one need only know a tabulated value of $k_{r,n}$ and an appropriate value for s; s is a function of r and n.

A corresponding estimator for a is then given by $a^{**}_{r,n} = X_{(s)} - E(Z_s)b^{**}_{r,n}$ where $Z_s = (X_{(s)} - a)/b$.

MSS, pp. 208-214, 241-252, give tables and references to additional tables for using these estimators. More recent references that aid in the use of these estimators are in Engelhardt (1975).

The estimators $b^{**}_{r,n}$ and $a^{**}_{r,n}$ approximate the BLU estimators and can be converted easily to approximations to the BLI estimators, which in turn approximate results obtained by maximum likelihood procedures.

The estimator $b^{**}_{r,n}$ has the property that $2b^{**}_{r,n}/\mathrm{var}(b^{**}_{r,n}/b)$ is very nearly a chi-squared variate with $2/\mathrm{var}(b^{**}_{r,n}/b)$ degrees of freedom. This property holds for any efficient unbiased estimator of b, including a maximum likelihood estimator corrected for bias. Because the BLI estimators so closely approximate the maximum likelihood estimators of b, tables yielding biases for the BLI estimators can be used to correct the maximum likelihood estimators for bias.

The fact that unbiased estimators of b are approximately chi-squared variates has been used to find approximations to the sampling distributions of functions of estimators of a and other distribution percentiles. MSS describe an F-approximation that can be used with complete samples to obtain confidence bounds on very high (above or below 90 percent), or very low distribution percentiles, or with highly censored data to obtain a confidence bound for a. The precision of this approximation is discussed by Lawless (1975) and Mann (1977, 1978). Engelhardt and Bain (1977) have suggested the use of a $\ell n\ \chi^2$ approximation, the regions of utility of which tend to complement those of the F-approximation. Lawless (1978) reviews methods for constructing confidence intervals or other characteristics of the Weibull or extreme-value distribution.

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